A Natural Framing for Asymptotically Flat Integral Homology 3-Sphere

Su-Win Yang

Communicated with W. H. Lin

Abstract

For an integral homology 3-sphere embedded asymptotically flatly in an Euclidean space, we find a natural framing extending the standard trivialization on the asymptotically flat part.

Suppose $\overline{M}$ is a 3-dimensional closed smooth manifold which has the same integral homology groups as the 3-sphere S^3. x_0 is a fixed point in $\overline{M}$. Embed $\overline{M}$ in a Euclidean space $\mathbb{R}^n$ such that x_0 is the infinite point of the 3-dimensional flat space $\mathbb{R}^3 \times \{0\}$ of $\mathbb{R}^n$ and a neighborhood of x_0 contains the whole flat space $\mathbb{R}^3 \times \{0\}$ except a compact set. Precisely, for any positive number r, let B_r denote the closed ball of radius r in $\mathbb{R}^3$ and $N_r = (\mathbb{R}^3 - B_r) \times \{0\}$; there exists r_0, a positive number, such that N_{r_0} is contained in $\overline{M}$ and $N_{r_0} \cup \{x_0\}$ is an open neighborhood of x_0 in $\overline{M}$.

Let $M = \overline{M} - \{x_0\}$, it is an asymptotically flat 3-dimensional manifold with acyclic homology. The main purpose of this article is to define a natural framing for M. If we identify the tangent spaces of points in the flat part N_{r_0} with $\mathbb{R}^3 \times \{0\}$, then the tangent bundle of M can be thought as a 3-dimensional vector bundle over the closed manifold $M_0 = M/\overline{N}_s$, where s is a number greater than r_0 and $\overline{N}_s$ is the closure of N_s; we shall call this vector bundle the tangent bundle $T(M_0)$ of M_0. And our natural framing is just a trivialization of $T(M_0)$, which corresponds to a trivialization of the tangent bundle $T(M)$ whose restriction to the flat part is the standard trivialization on $\mathbb{R}^3$. Because M_0 is a closed 3-manifold, there are countably infinite many

choices of framings associated with the infinite elements in $[M_0, SO(3)]$. (When $H_*(M_0) \approx H_*(S^3)$, $[M_0, SO(3)] \approx [S^3, SO(3)] \approx \mathbf{Z}$.) Therefore, our natural framing is a special choice from the infinite many.

On the other hand, this natural framing for $T(M_0)$ can also provide a special one-to-one correspondence between the infinite framings of S^3 and that of $\overline{M}$. (Note: Here, we do not think that $\overline{M}$ and M_0 have the same tangent bundle. Conversely, we may think that the tangent bundle of $\overline{M}$ is equal to the connected sum of the tangent bundles of M_0 and S^3.)

There are two main steps to the natural framing on $T(M_0)$.

Step 1 A special map from $C_2(M)$ to S^2

We define $C_2(M)$ at first.

For any set X, $\Delta(X)$ denote the diagonal subset $\{(x,x) \in X \times X, x \in X\}$ of $X \times X$ and $C_2(X) = X \times X - \Delta(X)$. Thus $C_2(M)$ is the configuration space of all pairs of distinct two points in M.

Fix some large number s such that $M \subset (B_s \times \mathbb{R}^{n-3}) \cup N_s$.

For any $r \geq s$, let $B_r = \{x \in \mathbb{R}^3 : |x| \leq r\}$, $N_r = (\mathbb{R}^3 - B_r) \times \{0\}$ and $M_r = M - N_r$.

Let Y denote the union of the following three subsets of $C_2(M)$:

$$\text{(i)} \quad Y_0 = C_2(N_s)$$

$$\text{(ii)} \quad Y_1 = \cup_{r \geq s}(N_{r+s} \times M_r)$$

$$\text{(iii)} \quad Y_2 = \cup_{r \geq s}(M_r \times N_{r+s})$$

Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^3$ denote the projection

$$\pi(t_1, t_2, \cdots, t_n) = (t_1, t_2, t_3)$$

and $f : Y \longrightarrow S^2$ denote the map

$$f(x, y) = \frac{\pi(y - x)}{|\pi(y - x)|}$$

for $(x, y) \in Y$, $x, y \in M$.

For the well-defining of the map f, we should check that $|\pi(y - x)|$ is a non-zero value. When (x, y) is in Y_0, $|\pi(y - x)| = |y - x|$, it is non-zero. When (x, y) is in Y_1, (x, y) is in $N_{r+s} \times M_r$ for some $r \geq s$; thus $\pi(x)$ is outside of B_{r+s} and $\pi(y)$ is in B_r, and hence $\pi(y - x) = \pi(y) - \pi(x)$, it has also a non-zero norm. It is similar for the case that (x, y) is in Y_2.

The following proposition describes some homology properties for the space Y and the map f.

Proposition 1

(i) $H_*(Y) \approx H_*(S^2)$

(ii) $f_*; H_2(Y) \longrightarrow H_2(S^2)$ is an isomorphism.

(iii) Let $j : Y \longrightarrow C_2(M)$ denote the inclusion map.

$$j_* : H_i(Y) \longrightarrow H_i(C_2(M))$$

is isomorphic, for all integer $i \geq 0$. ∎

In the proof of the proposition, we strongly use the assumption that $H_*(M)$ is acyclic.

Remark: All the homologies in this article are with integral coefficients.

By Proposition 1, the continuous map $f : Y \longrightarrow S^2$ uniquely extends to a continuous map $\overline{f} : C_2(M) \longrightarrow S^2$ up to homotopy relative to the subspace Y. (That is, if both $\overline{f}_1$ and $\overline{f}_2$ are the extensions of f to the whole space $C_2(M)$, then there is a homotopy $F : C_2(M) \times [0, 1] \longrightarrow S^2$ such that

$F(\xi, 0) = \overline{f}_1(\xi)$, $F(\xi, 1) = \overline{f}_2(\xi)$, for all $\xi \in C_2(M)$, and $F(\xi', t) = f(\xi')$ for all $\xi' \in Y$ and $t \in [0, 1]$.)

Usually, the homotopy class of a map from $C_2(M)$ to S^2 can not give any framing on $T(M_0)$. But the extension of f does give a framing on $T(M_0)$ as shown in Step 2.

Step 2 The framing determined by the map $\overline{f}$ on $C_2(M)$

The normal bundle of $\Delta(M)$ in $M \times M$ can be identified as the tangent bundle $T(M)$ of M. Consider a suitable compactification of $C_2(M)$, the spherical bundle $S(TM)$ become a part of boundary of $C_2(M)$. Let $h : S(TM) \longrightarrow S^2$ denote the restriction of $\overline{f}$ to $S(TM)$. On the flat part N_s of M, the spherical bundle $S(TN_s) = N_s \times S^2$ and h on $S(TN_s)$ is equal to the map restricted from f which is exactly the projection from $N_s \times S^2$ to S^2. Thus h induces a map $h_0 : S(TM_0) \longrightarrow S^2$.

$S(TM_0)$ is a $SO(3)$-bundle over M_0.

Can $h_0 : S(TM_0) \longrightarrow S^2$ determine uniquely an orthogonal map, that is, a fibrewise orthogonal map? (An orthogonal map is exactly a framing for the vector bundle.) There is also an interesting question that can h_0 be homotopic to an orthogonal map; if such an orthogonal map exists, is it unique up to homotopy? We shall answer the questions partially.

Choose a framing for $S(TM_0)$ and we may think h_0 as a map from $M_0 \times S^2$ to S^2. Let y_0 denote the point in M_0 representing the set N_s. Then the restriction of h_0 to $y_0 \times S^2$ is the identity map of S^2. Thus the restriction of h_0 to each fibre $x \times S^2$, $x \in M_0$, is also a homotopy equivalence; and hence, h_0 induces a map $\widehat{h}_0$ from M_0 to $G(3)$, the space of all homotopy equivalences of S^2 to itself. Choose a base point z_0 in S^2, and consider the subspace $F(3)$ of $G(3)$ consisting of all the homotopy equivalences which fix the base point z_0. Then $F(3)$ is the fibre of the fibration $G(3)$ over S^2, it is the key fact for the homotopic computations.

For any two spaces X_1 and X_2 with base points x_1 and x_2, respectively, $[X_1, X_2]$ denotes the set of homotopy classes of continuous maps from X_1 to X_2 and sending x_1 to x_2. In the following, $M_0$is with base point y_0 representing the set $\overline{N}_s$; $SO(3)$, $G(3)$ and $F(3)$ are with the base point the identity of S^2. We shall consider only the maps sending the base point to base point and consider only the homotopies which keep the base point fixed.

M_0 has the same homology as S^3. Usually, we can not expect they also have the same homotopy behavior. But we still have the following proposition.

Proposition 2 Suppose $\phi : M_0 \longrightarrow S^3$ is a degree 1 map. Then the homotopy classes $[M_0, SO(3)]$, $[M_0, G(3)]$, $[M_0, F(3)]$ are all groups, and the group homomorphisms induced by ϕ,

$$[S^3, SO(3)] \xrightarrow{\phi^\sharp} [M_0, SO(3)]$$

$$[S^3, G(3)] \xrightarrow{\phi^\sharp} [M_0, G(3)]$$

$$[S^3, F(3)] \xrightarrow{\phi^\sharp} [M_0, F(3)]$$

$$[S^3, S^2] \xrightarrow{\phi^\sharp} [M_0, S^2]$$

are all isomorphisms of groups. ∎

There are further relations between these homotopy classes.

Proposition 3 Let $p : SO(3) \longrightarrow G(3)$ and $q : F(3) \longrightarrow G(3)$ denote the inclusions. Then, for any integral homology 3-sphere M_0, the homomorphism

$$p_* \oplus q_* : [M_0, SO(3)] \oplus [M_0, F(3)] \longrightarrow [M_0, G(3)]$$

is an isomorphism.
Especially, when $M_0 = S^3$, we have

$$\pi_3(G(3)) \approx \pi_3(SO(3)) \oplus \pi_3(F(3)) \ .$$

$\blacksquare$

Furthermore, the group isomorphism

$$q_*^{-1} : [M_0, G(3)]/p_*([M_0, SO(3)] \longrightarrow [M_0, F(3)]$$

induces a group homomorphism

$$Q : [M_0, G(3)] \longrightarrow [M_0, F(3)] \approx \mathbf{Z_2} \quad .$$

For a continuous map $g : M_0 \times S^2 \longrightarrow S^2$, let $\widehat{g}$ denote the map from M_0 to $G(3)$ defined by $\widehat{g}(x)(y) = g(x, y)$, for $x \in M_0$ and $y \in S^2$ and let $Q(g) = Q([\widehat{g}])$.

Theorem 4 A continuous map $g : M_0 \times S^2 \longrightarrow S^2$ is homotopic to an orthogonal map, if and only if, $Q(g) = 0$ in $[M_0, F(3)]$. $\blacksquare$

Now, h_0 still denotes the map from $S(TM_0)$ to S^2 given by the map $\overline{f} : C_2(M) \longrightarrow S^2$. Choose a framing for TM_0, $\psi : S(TM_0) \longrightarrow M_0 \times S^2$, it is a fibre map and fibrewise orthogonal. Then $h_0 \circ \psi^{-1}$ is a map from $M_0 \times S^2$ to S^2 and the value $Q(h_0 \circ \psi^{-1})$ is independent of the choice of the framing ψ. Therefore, $Q(h_0 \circ \psi^{-1})$ is an invariant of the integral homology 3-sphere $\overline{M}$, it is the obstruction for h_0 to be homotopic to an orthogonal map. We hope that this is not really an obstruction.

Conjecture 5 $Q(h_0 \circ \psi^{-1}) = 0$, for any integral homology 3-sphere $\overline{M}$. $\blacksquare$

On the other hand, the group isomorphism

$$p_*^{-1} : [M_0, G(3)]/q_*([M_0, F(3)] \longrightarrow [M_0, SO(3)]$$

induces a group homomorphism

$$P : [M_0, G(3)] \longrightarrow [M_0, SO(3)] \quad .$$

For a continuous map $g : M_0 \times S^2 \longrightarrow S^2$, let $P(g) = P([\widehat{g}])$.

For the map h_0 and the corresponding element $P(h_0 \circ \psi^{-1})$ in $[M_0, SO(3)]$, choose an orthogonal map $g_0 : M_0 \times S^2 \longrightarrow S^2$ such that the associated map $\widehat{g}_0$ is in the homotopy class $P(h_0 \circ \psi^{-1})$. Then we get an orthogonal map $g_0 \circ \psi : S(TM_0) \longrightarrow S^2$ which represents a homotopy class of framings determined by h_0, also by the map $\overline{f} : C_2(M) \longrightarrow S^2$. This framing can also be characterized by the following theorem.

Theorem 6 There exists a framing $\psi_0 : S(TM_0) \longrightarrow M_0 \times S^2$ unique up to homotopy such that $P(h_0 \circ \psi_0^{-1}) = 0$. ∎

Proofs

Outline of Proof of Proposition 1

N_s is a subset of $\mathbb{R}^3 \times \{0\}$. In N_s, we choose a subspace S_3 which is a deformation retract of N_s and a point x_1 in the bounded component of $\mathbb{R}^3 \times \{0\} - S_3$. Let $S = \{x_1\} \times S_3$, it is a subspace of Y. We show that the three maps, the inclusion of S in Y, the restriction of f to S, and the restriction of j to S, all induce isomorphisms of homology groups of the corresponding spaces. That is, $H_*(S) \longrightarrow H_*(Y)$, $(f|_S)_* : H_*(S) \longrightarrow H_*(S^2)$, and $(j|_S)_* : H_*(S) \longrightarrow H_*(C_2(M))$ all are isomorphisms.

Proof of Proposition 1

First we compute the homology of Y_0, Y_1, Y_2, separately.

$Y_0 = C_2(N_s) = N_s \times N_s - \Delta(N_s) \subset N_s \times N_s$. N_s is homeomorphic to $S^2 \times (s, \infty)$. Thus $H_*(N_s \times N_s) \approx H_*(S^2 \times S^2)$. By Thom Isomorphism, $H_i(N_s \times N_s, Y_0) \approx H_{i-3}(N_s)$.

Now, we use the long exact sequence of the pair $(N_s \times N_s, Y_0)$ to determine $H_*(Y_0)$.

$$\longrightarrow H_{i+1}(N_s \times N_s, Y_0) \xrightarrow{\partial_*} H_i(Y_0) \longrightarrow H_i(N_s \times N_s) \longrightarrow$$
$$\longrightarrow H_i(N_s \times N_s, Y_0) \longrightarrow \cdots$$

When i is odd, both $H_{i+1}(N_s \times N_s, Y_0)$ and $H_i(N_s \times N_s)$ are the trivial group $\{0\}$. Thus we have

$$H_4(Y_0) \approx H_4(N_s \times N_s) \oplus \partial_*(H_5(N_s \times N_s, Y_0) \approx \mathbf{Z} \oplus \mathbf{Z}$$

$$H_2(Y_0) \approx H_2(N_s \times N_s) \oplus \partial_*(H_3(N_s \times N_s, Y_0) \approx \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$$

and $H_i(Y_0)$ is trivial, if i is odd.

($\mathbf{Z}$ denotes the group of integers.)

To find the generators of H_4 and H_2 of Y_0, we choose three 2-spheres S_1, S_2, S_3 in $\mathbb{R}^3 \times \{0\}$ of radius $2s, 4s, 6s$, respectively, all with center the origin. (S_i is the boundary of $N_{2s\times i}$, $i = 1, 2, 3$.) For each i, $i = 1, 2, 3$, choose a point x_i in S_i. The 2-spheres are also oriented in the same way, that is, the natural diffeomorphisms of the 2-spheres are orientation-preserving. Then $S_i \times x_j$ and $x_j \times S_i$, $1 \leq i \neq j \leq 3$, are 2-cycles in Y_0, also in $N_s \times N_s$; $S_i \times S_j$, $1 \leq i \neq j \leq 3$, are 4-cycles in Y_0, also in $N_s \times N_s$.

In the following, if c is a cycle in Y_0, $[c]$ shall denote the corresponding homology class in Y_0.

Lemma 7

(i) $[S_1 \times S_3]$ is the generator of $H_4(N_s \times N_s)$.

(ii) $[(S_1 - S_3) \times S_2]$ is the generator of the subgroup $\partial_*(H_5(N_s \times N_s, Y_0))$ in $H_4(Y_0)$. ∎

We use the lemma to prove Proposition 1, and prove the lemma later.

There are some relations between these classes in $H_*(Y_0)$:
$[(S_1 - S_3) \times S_2] = [S_1 \times S_2] - [S_3 \times S_2]$, $\quad [(S_1 \times S_2] = [S_1 \times S_3]$
and $[S_3 \times S_2] = [S_3 \times S_1]$.

Thus $[S_1 \times S_3]$ and $[S_3 \times S_1]$ form the basis of $H_4(Y_0)$.

Similarly, $[S_1 \times x_3]$ and $[x_1 \times S_3]$ are the basis of $H_2(N_s \times N_s)$; $[(S_1 - S_3) \times x_2]$ ($= \epsilon_0[x_2 \times (S_1 - S_3)]$, ϵ_0 is 1 or -1) is the generator of the subgroup $\partial_*(H_3(N_s \times N_s, Y_0)$ in $H_2(Y_0)$.

Thus $[S_1 \times x_3]$, $[x_1 \times S_3]$ and $[(S_1 - S_3) \times x_2]$ form a basis of $H_2(Y_0)$.

Now we study the homology of Y_1 and Y_2.

It is easy to see that the inclusion of $N_{4s} \times M_{3s}$ in Y_1 and the inclusion of $S_3 \times M_{3s}$ in $N_{4s} \times M_{3s}$ both are homotopy equivalences. Thus $H_*(Y_1) \approx H_*(S_3 \times M_{3s}) \approx H_*(S_3)$. (Recall: M_r is acyclic, for any $r \geq s$.) Similarly, Y_2 also has the same homology as 2-sphere.

Y_1 and Y_2 are disjoint, and hence the homology of their union $Y_1 \cup Y_2$ is also determined. We can use the Mayer-Vietoris Sequence of the triple $(Y, Y_0, Y_1 \cup Y_2)$ to find the homology of Y. In fact, we have

(i) The cycle $S_1 \times S_3$ is contained in Y_2 and is killed in Y_2.

(ii) The cycle $S_3 \times S_1$ is contained in Y_1 and is killed in Y_1.

(iii) The cycle $x_3 \times S_1$ is contained in Y_1 and is killed in Y_1.

(iv) The cycle $S_1 \times x_3$ is contained in Y_2 and is killed in Y_2.

Therefore, $H_4(Y) = \{0\}$ and in $H_2(Y)$, we have $[x_1 \times S_3]$ and $[S_3 \times x_2]$ left; the equality $[(S_1 - S_3) \times x_2] = \epsilon_0[x_2 \times (S_1 - S_3)]$ become the new equality $-[S_3 \times x_2] = -\epsilon_0[x_2 \times S_3]$. Thus $[x_1 \times S_3] = [x_2 \times S_3] = \epsilon_0[S_3 \times x_2] = \epsilon_0[S_3 \times x_1]$. This proves that $H_*(Y) \approx H_*(S^2)$. Actually, we know more than that: the inclusion of the space $\{x_1\} \times S_3$ in Y induces isomorphisms of the homology groups. It is easy to see that the map f, restricted to $\{x_1\} \times S_3$, is an homotopy equivalence from $\{x_1\} \times S_3$ to S^2. This proves the second statement that f_* is an isomorphism.

To prove the third statement that j_* is an isomorphism, it is also enough to show that the restriction of j to $\{x_1\} \times S_3$ induces isomorphisms for the

homology groups. Similar to the computation of the homology of $C_2(N_s)$, we consider the long exact sequence of pair $(M \times M, C_2(M))$

$$H_{i+1}(M \times M) \longrightarrow H_{i+1}(M \times M, C_2(M)) \xrightarrow{\partial_*} H_i(C_2(M)) \\ \longrightarrow H_i(M \times M) \longrightarrow .$$

Because $H_*(M)$ is acyclic, $H_*(M \times M)$ is also acyclic.

We have

$$H_i(C_2(M)) \approx H_{i+1}(M \times M, C_2(M)), \text{ for all } i \geq 1 \ .$$

But $H_{i+1}(M \times M, C_2(M)) \approx H_{i-2}(M)$, by the Thom Isomorphism. Thus $C_2(M)$ has the same homology as 2-sphere.

And it is easy to see that the inclusion of $\{x_1\} \times (M, M - x_1)$ in $(M \times M, C_2(M))$ induces isomorphisms of homology groups, and hence, the inclusion of $\{x_1\} \times (M - x_1)$ in $C_2(M)$ also induces isomorphisms of homology groups. The cycle $\{x_1\} \times S_3$ is a generator of $H_2(\{x_1\} \times (M - x_1))$, and hence also a generator of $H_2(C_2(M))$. This proves the third statement that j_* is an isomorphism.

(i) of Lemma 7 is obvious. Now, we are going to prove (ii) in Lemma 7.

Consider the following commutative diagram

$$\begin{array}{ccc}
H_2(S_2) \otimes H_3(N_s, N_s - S_2) & \xrightarrow{id \otimes \partial_*} & H_2(S_2) \otimes H_2(N_s - S_2) \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \eta_1} \\
H_5(S_2 \times N_s, S_2 \times N_s - S_2 \times S_2) & \xrightarrow{\partial_*} & H_4(S_2 \times N_s - S_2 \times S_2) \\
\downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \eta_2} \\
H_5(S_2 \times N_s, S_2 \times N_s - \Delta(S_2)) & \xrightarrow{\partial_*} & H_4(S_2 \times N_s - \Delta(S_2)) \\
\downarrow{\scriptstyle \tau_3} & & \downarrow{\scriptstyle \eta_3} \\
H_5(N_s \times N_s, Y_0) & \xrightarrow{\partial_*} & H_4(Y_0)
\end{array}$$

The maps τ_1 and η_1 are isomorphisms from Kunneth formula. Other homomorphisms are induced by the corresponding inclusion maps. τ_2 is an isomorphism by the result of Lefschetz Duality in the 5-dimensional manifold $S_2 \times N_s$; τ_3 is an isomorphism by the result of Thom Isomorphism Theorem. Precisely, consider the following commutative diagram

$$\begin{array}{ccc}
H_5(S_2 \times N_s, S_2 \times N_s - S_2 \times S_2) & \xrightarrow{\sigma_1} & H^0(S_2 \times S_2) \\
\downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_4} \\
H_5(S_2 \times N_s, S_2 \times N_s - \Delta(S_2)) & \xrightarrow{\sigma_2} & H^0(\Delta(S_2))
\end{array}$$

where $\sigma_i, i = 1, 2$, are the isomorphisms of Lefschetz Duality, τ_4 is the homomorphism induced by the inclusion.

Because τ_4 is an isomorphism, τ_2 is also an isomorphism. The proof of isomorphism of τ_3 is in some sense analogous to that for τ_2, we omit it.

¿From the long exact sequence of the pair $(N_s, N_s - S_2)$, it is easy to see that $[S_1 - S_3]$ is the generator of $\partial_*(H_3(N_s, N_s - S_2))$, and hence, $[S_2 \times (S_1 - S_3)]$

is the generator of $(id\otimes\partial_*)(H_2(S_2)\otimes H_3(N_s, N_s-S_2))$. By the commutativity of the above diagram, $[S_2\times(S_1-S_3)]$ $(=-[(S_1-S_3)\times S_2])$ is the generator of $\partial_*(H_5(N_s\times N_s, Y_0))$. This proves Lemma 7 and completes the long proof of **Proposition 1**.

Proof of Proposition 2

We need to show the isomorphisms between $[M_0, X]$ and $[S^3, X]$, for $X = SO(3), G(3), F(3)$ and S^2.

For the case of $SO(3)$, we consider the classifying space $BSO(3)$ of the $SO(3)$-bundles. Then

$$[M_0, SO(3)] \approx [SM_0, BSO(3)] \text{ and } [S^3, SO(3)] \approx [S^4, BSO(3)] \ ,$$

where SM_0 is the suspension of M_0. On the other hand, because SM_0 is simply connected and the map $S(\phi) : SM_0 \longrightarrow SS^3(= S^4)$ induces isomorphisms of homology groups, $S(\phi)$ is a homotopy equivalence. Thus $S(\phi)^\sharp$:$[SM_0, BSO(3)] \longrightarrow [S^4, BSO(3)]$ is isomorphic, and hence,

$$[M_0, SO(3)] \approx [S^3, SO(3)] \ .$$

For the cases of $G(3)$ and $F(3)$, we may also consider the corresponding classifying spaces, by the result of Fuchs [2]; and the proof is completely similar.

The group property of the associated homotopy classes is a result of Dold and Lashof [1]; for the convenience of interested reader, we give a proof in the appendix.

For the case of S^2, it is enough to note that $[M_0, S^2] \approx [M_0, S^3]$ ($\approx H^3(M_0)$), which implies the isomorphism we need.

Proof of Proposition 3

By Proposition 2, it is enough to prove the result for the case that $M_0 = S^3$.

Consider the commutative diagram of fibrations over S^2

$$\begin{array}{ccccc} S^1 & \longrightarrow & SO(3) & \longrightarrow & S^2 \\ \downarrow & & \downarrow p & & \downarrow id \\ F(3) & \xrightarrow{q} & G(3) & \longrightarrow & S^2 \end{array}$$

and the associated commutative diagram of exact sequences of homotopy groups

$$\begin{array}{ccccc} \pi_i(S^1) & \longrightarrow & \pi_i(SO(3)) & \longrightarrow & \pi_i(S^2) \\ \downarrow & & \downarrow p_* & & \downarrow id \\ \pi_i(F(3)) & \xrightarrow{q_*} & \pi_i(G(3)) & \xrightarrow{\alpha} & \pi_i(S^2) \end{array}$$

For $i \geq 3$, $\pi_i(S^1) = \pi_{i-1}(S^1) = \{0\}$, and hence

$$\pi_i(SO(3)) \approx \pi_i(S^2) \quad .$$

Thus $p_* : \pi_i(SO(3)) \longrightarrow \pi_i(G(3))$ can be thought as the right-inverse of $\alpha : \pi_i(G(3)) \longrightarrow \pi_i(S^2)$. This implies that α is an epimorphism, q_* is a monomorphism, and p_* supplies the necessary homomorphism for splitting. Therefore,

$$\pi_i(G(3)) = q_*(\pi_i(F(3)) \oplus p_*(\pi_i(SO(3)), \text{ for all } i \geq 3$$

Appendix

The proof of the appendix is essentially from the proof of the main result in Dold and Lashof [1]. The author just write it for self-interesting.

Suppose H is a path-connected space and has an associative multiplication which has a two-sided unit e. For $h_1, h_2 \in H$, $h_1 \cdot h_2$ denotes the product of h_1 and h_2. Thus $h \cdot e = e \cdot h = h$, for all $h \in H$. Furthermore, assume X is a polyhedron. The purpose of this appendix is to show that the homotopy classes in $[X, H]$ form a group under the following multiplication:

For any two maps $f, g : X \longrightarrow H$, $(f \cdot g)(x) = f(x) \cdot g(x)$.

The associative law of this multiplication in $[X, H]$ is obvious. It is enough to show that for any $f : X \longrightarrow H$, there is a map $g : X \longrightarrow H$ such that $f \cdot g$ is homotopic to the constant map $\overline{e} : X \longrightarrow H$, $\overline{e}(x) = e$, for all $x \in X$.

We shall construct the map $g : X \longrightarrow H$ and the homotopy $D : X \times I \longrightarrow H$ satisfying $D(x, 0) = e$, $D(x, 1) = f(x) \cdot g(x)$, inductively on the skeleton of X. (I is the unit interval $[0, 1]$.)

$X^{(k)}$ denotes the k-skeleton of X.

Assume g is defined on $X^{(k)}$ and D is defined on $X^{(k)} \times I$ such that $D(x, 0) = e$ and $D(x, 1) = f(x) \cdot g(x)$, for all $x \in X^{(k)}$. If necessary, we may ask that the base point x_0 of X is in $X^{(0)}$ and $f(x_0) = g(x_0) = D(x_0, t) = e$, for any $t \in I$.

For any $(k+1)$-simplex Δ in $X^{(k+1)}$, we want to extend g to the part Δ and D to the part $\Delta \times I$. Let S denote the boundary of Δ, it is a k-sphere. S is in $X^{(k)}$, g is defined on S and D is defined on $S \times I$.

Claim $g|_S : S \longrightarrow H$ is null-homotopic.

Proof Δ is a simplex, there is a contraction map $\gamma : \Delta \times I \longrightarrow \Delta$, $\gamma(x, 0) = x$ and $\gamma(x, 1) = x_1$, for all $x \in \Delta$. x_1 is some fixed point in S. Let $\beta : S \times I \longrightarrow H$ denote the map $\beta(x, t) = f(\gamma(x, t)) \cdot g(x)$, for $x \in S$. Let

$y_1 = f(x_1)$ and $\overline{y}_1 : S \longrightarrow H$ denote the constant map sending the points of S to y_1. Then β is a homotopy between $f \cdot g$ and $\overline{y}_1 \cdot g$ on S. H is path-connected, $\overline{y}_1 \cdot g$ is homotopic to $\overline{e} \cdot g = g$. Thus g is homotopic to $f \cdot g$ on S. On the other hand, the restriction of D to $S \times I$ provides a homotopy between the restrictions of $f \cdot g$ and $\overline{e}$. This proves that $g|_S$ is null-homotopic.

Therefore, we can extend $g|_S$ to the part Δ, say, $g' : \Delta \longrightarrow H$, and we can also extend $D|_{S\times I}$ to the whole boundary of $\Delta \times I$ as follows:

We use $D' : \partial(\Delta \times I) \longrightarrow H$ to denote the extension. $\partial(\Delta \times I) = \Delta \times \{0\} \cup \Delta \times \{1\} \cup S \times I$.

$D'(x, 0) = e$ and $D'(x, 1) = f(x) \cdot g'(x)$, for all $x \in \Delta$;

$D'(y, t) = D(y, t)$, for all $y \in S$ and $t \in I$.

The map D' may not be extended to $\Delta \times I$. We shall find a map $g_1 : \Delta \longrightarrow H$ with $g_1|_S = \overline{e}|_S$ and modify the map D' by multiplying D' with g_1 on the part $\Delta \times \{1\}$ such that the new map is null-homotopic. Precisely, let $D'' : \partial(\Delta \times I) \longrightarrow H$ denote the map, $D''(\xi) = D'(\xi)$, for all $\xi \in \Delta \times \{0\} \cup S \times I$, $D''(x, 1) = D'(x, 1) \cdot g_1(x)$, for all $(x, 1) \in \Delta \times \{1\}$.

We may think the map g_1 as a map on $\Delta \times \{1\}$ and extend it trivially to the whole boundary $\partial(\Delta \times I)$, that is, sending all points undefined to e. Then D'' is just equal to $D' \cdot g_1$. To let D'' be null-homotopic, we can choose g_1 such that $[g_1] = [D']^{-1}$ in $\pi_{k+1}(H)$. Of course, g' should be changed to the new map $g' \cdot g_1$. Therefore, D'' is null-homotopic and its extension to $\Delta \times I$ also gives the homotopy between $f \cdot (g' \cdot g_1)$ on Δ. This finishes the extension of g and D to Δ.

References

[1] A. Dold and R. Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles*, Illi. J. Math. **3** (1959), 285-305.

[2] M. Fuchs, , *Verallgemeinerte Homotopie-Homomorphismen und klassi-*

fizierende Raume, Math. Ann. **161**, (1965), 197-230.

Department of Mathematics
National Taiwan University
Taipei, Taiwan
E-mail: swyang@math.ntu.edu.tw